Week 9 Indefinite integral

For a function $f(x)$

with respect to x

$\int f(x)\,dx$ = the indefinite integral of $f(x)$

integration sign = anti-derivative of $f(x)$

$$g'(x) = f(x) \iff g(x) = \int \underbrace{f(x)}_{\text{called integrand}}\,dx$$

integration variable is x

f is the derivative of g

g is an anti-derivative of f

eg $(\tan^{-1}x)' = \dfrac{1}{1+x^2}$. Also, $(\tan^{-1}x + 1)' = \dfrac{1}{1+x^2}$

$\therefore$ Both $\tan^{-1}x$ and $\tan^{-1}x + 1$ are anti-derivative of $\dfrac{1}{1+x^2}$

$\tan^{-1}x = \arctan x$

not $(\tan x)^{-1} = \dfrac{1}{\tan x}$

In general,

$(\tan^{-1}x + C)' = \dfrac{1}{1+x^2}$ for any constant C

$\therefore \displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$, where C is a constant

called integration constant

eg Verify that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

Sol Need to show $(\ln|x| + C)' = \dfrac{1}{x}$

For $x > 0$, $|x| = x$.

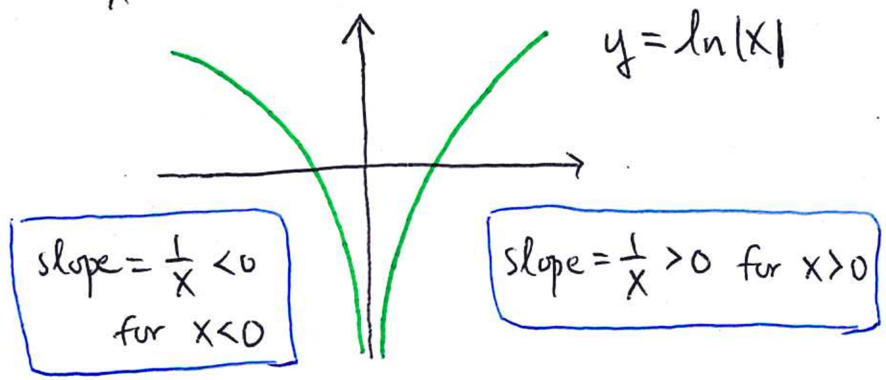
$\therefore (\ln|x| + C)' = (\ln x + C)' = \dfrac{1}{x}$

For $x < 0$, $|x| = -x$,

$\therefore (\ln|x| + C)' = (\ln(-x) + C)'$

$$= \underbrace{\frac{1}{-x}\cdot(-1)}_{\text{Chain rule}} = \frac{1}{x}$$

$\therefore \displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$



Some basic integrals (k, a, b are constants)

$\int (af(x)+bg(x))dx = a\int f(x)dx + b\int g(x)dx$

$\int k\,dx = kx + C$

$\int x^k dx = \frac{1}{k+1}x^{k+1} + C$ for $k \neq 1$

$\int \frac{1}{x}dx = \ln|x| + C$

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{1}{\ln a}a^x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \csc x \cot x\,dx = -\csc x + C$

$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$

$\int \frac{1}{1+x^2}dx = \arctan x + C$

$\int \frac{1}{|x|\sqrt{x^2-1}}dx = \operatorname{arcsec} x + C$

Recall: $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

Q $\int \frac{-1}{\sqrt{1-x^2}}dx = \arccos x + C$?

or $-\arcsin x + C$?

A Both are correct!

$\underbrace{\arccos x}\ = \underbrace{-\arcsin x}\ + \frac{\pi}{2}$

differed by a constant

eg $\int \left(4x^2 - \csc^2 x - \frac{1}{1+x^2}\right)dx$

$= 4\int x^2 dx - \int \csc^2 x\,dx - \int \frac{1}{1+x^2}dx$

$= \frac{4}{3}x^3 + \cot x - \arctan x + C$

eg Suppose $f'(x) = x^3 - 1$, $f(2) = 1$. Find $f(x)$

Sol $f'(x) = x^3 - 1 \Rightarrow f(x) = \int (x^3-1)dx$

$= \frac{1}{4}x^4 - x + C$

$f(2) = 1 \Rightarrow \frac{1}{4}(2)^4 - 2 + C = 1$

$4 - 2 + C = 1$

$C = -1$

$\therefore f(x) = \frac{1}{4}x^4 - x - 1$

Integration by substitution

Let $f(u)$ be a function of u

$u=u(x)$ be a function of x

Then $\underbrace{\int f(u(x))\frac{du}{dx}\,dx}_{\text{in terms of } x} = \underbrace{\int f(u)\,du}_{\text{in terms of } u}$

Rmk It can be proved from Chain rule

eg $\int \sqrt{3x+4}\,dx$

Sol let $u=3x+4$, $\frac{du}{dx}=3$, $du=3dx$

$$\int\sqrt{3x+4}\,dx = \frac{1}{3}\int\sqrt{u}\,du$$
$$= \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$$
$$= \frac{2}{9}(3x+4)^{\frac{3}{2}} + C$$

eg $\int e^{2x^2+1}\,x\,dx$

let $u=2x^2+1$ $\quad \frac{du}{dx}=4x \quad du=4x\,dx$

$$\therefore \int e^{2x^2+1}x\,dx = \frac{1}{4}\int e^u\,du$$
$$= \frac{1}{4}e^u + C$$
$$= \frac{1}{4}e^{2x^2+1} + C$$

Rmk

$\int e^{2x^2+1}\,dx$ is more difficult

eg $\int \frac{(1+\ln x)^6}{x}\,dx$

let $u=1+\ln x$, $\frac{du}{dx}=\frac{1}{x}$ $\quad du=\frac{1}{x}dx$

$$\therefore \int\frac{(1+\ln x)^6}{x}\,dx = \int u^6\,du = \frac{1}{7}u^7 + C$$
$$= \frac{1}{7}(1+\ln x)^7$$

eg $\int \frac{dx}{2x+1} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C$

let $u=2x+1$ $\quad du=2dx$ $\quad = \frac{1}{2}\ln|2x+1| + C$

eg $\int \cot(kx)\,dx$, where $k \neq 0$

Sol $\int \cot(kx)\,dx = \int \frac{\cos(kx)}{\sin(kx)}\,dx$

let $u = \sin(kx)$, then $du = k\cos(kx)\,dx$

$$\int \cot(kx)\,dx = \frac{1}{k}\int \frac{du}{u}$$

$$= \frac{1}{k}\ln|u| + C$$

$$= \frac{1}{k}\ln|\sin(kx)| + C$$

Faster way of writing this!

$$\int \cot(kx)\,dx = \int \frac{\cos(kx)}{\sin(kx)}\,dx = \frac{1}{k}\int \frac{\cos(kx)}{\sin(kx)}\,d(kx)$$

$$= \frac{1}{k}\int \frac{d\sin(kx)}{\sin(kx)}$$

$$= \frac{1}{k}\ln|\sin(kx)| + C$$

Ex Show $\int \tan(kx)\,dx = \ln|\sec(kx)| + C$

Trig formula

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$A = B \Rightarrow \begin{cases} \sin A\cos A = \frac{1}{2}\sin 2A \\ \cos^2 A = \frac{1}{2}(1+\cos 2A) \\ \sin^2 A = \frac{1}{2}(1-\cos 2A) \end{cases}$$

R.H.S is easier for integration

eg $\int \sin 5x \cos 3x\,dx$

$$= \int \frac{1}{2}(\sin 8x + \sin 2x)\,dx$$

$$= \frac{1}{16}\int \sin 8x\,d(8x) + \frac{1}{4}\int \sin 2x\,d(2x)$$

$$= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$$

eg $\int \cos x \cos^2 3x \, dx$

$= \int \cos x \left(\frac{1}{2}(1+\cos 6x) \right) dx$

$= \frac{1}{2} \int (\cos x + \cos x \cos 6x) \, dx$

$= \frac{1}{2} \sin x + \frac{1}{2} \int \frac{1}{2} \left[\cos 7x + \cos(-5x) \right] dx$

$= \frac{1}{2} \sin x + \frac{1}{4} \int (\cos 7x + \cos 5x) \, dx$

$= \frac{1}{2} \sin x + \frac{1}{28} \sin 7x + \frac{1}{20} \sin 5x + C$

Some product of trig. functions

1. $\int \sin^m x \cos^n x \, dx$

Case I: m is odd

Let $u = \cos x$. Then $\sin^2 x = 1 - \cos^2 x$

$= 1 - u^2$

$\sin x \, dx = -d\cos x$

$= -du$

eg. $\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx$

$= \int (1-u^2)^2 (-du)$

$= -\int (1 - 2u^2 + u^4) \, du$

$= -u + \frac{2}{3} u^3 - \frac{1}{5} u^5 + C$

$= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$

Case II: n is odd

Let $u = \sin x$. Then $\cos^2 x = 1 - \sin^2 x$

$= 1 - u^2$

$\cos x \, dx = d \sin x$

$= du$

eg $\int \sin^3 x \cos^3 x \, dx$

$= \int \sin^3 x \cos^2 x \cos x \, dx$

$= \int \sin^3 x (1 - \sin^2 x) \, d\sin x$

$= \int (\sin^3 x - \sin^5 x) \, d\sin x$

$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$

Rmk It satisfies both Case I and II
Both methods work

Case III : Both m, n are even (More difficult)

Apply formulas $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$

to reduce power

eg $\int \sin^4 x \cos^2 x \, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot \frac{1+\cos 2x}{2} \, dx$

$= \frac{1}{8}\int (1-\cos 2x - \cos^2 2x + \cos^3 2x) dx$

$= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2} dx + \frac{1}{8}\underbrace{\int \cos^3 2x \, dx}_{\text{Case II (Try it!)}}$

$= \frac{1}{8}x - \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{64}\sin 4x$

$\quad + \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + C$

$= \frac{1}{8}x - \frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + C$

2. $\int \tan^m x \sec^n x \, dx$

$\boxed{\begin{aligned} &1+\tan^2 x = \sec^2 x \\ &u = \sec x \\ &du = \sec x \tan x \, dx \\ &d\sec x = \sec x \tan x \, dx \end{aligned}}$

Case I : odd $m = 2k+1$

$\int \tan^{2k+1} x \sec^n x \, dx$

$= \int (\tan^2 x)^k \sec^{n-1} x \, d\sec x$

$= \int (u^2-1)^k u^{n-1} du$

Case II : even $n = 2k$

$\boxed{\begin{aligned} &u = \tan x \\ &du = \sec^2 x dx \\ &d\tan x = \sec^2 x dx \end{aligned}}$

$\int \tan^m x \sec^{2k} x \, dx$

$= \int \tan^m x \sec^{2k-2} x \, d\tan x$

$= \int u^m (1+u^2)^{k-1} du$

Case III : m is even, n is odd

"Integration by parts"

Dissuss later

Trig. Substitution

eg. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ with $x = 3\sin\theta$

Sol let $x = 3\sin\theta$ $\quad dx = 3\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int \frac{3\cos\theta\, d\theta}{3\cos\theta}$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arcsin\left(\frac{x}{3}\right) + C$$

Generally, if a is a positive constant

For $\sqrt{a^2-x^2}$, try $x = a\sin\theta$

For $\sqrt{x^2-a^2}$, try $x = a\sec\theta$

For $\sqrt{a^2+x^2}$, try $x = a\tan\theta$

eg $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$ $\qquad$ let $x = a\sec\theta$

$\qquad dx = a\sec\theta\tan\theta\, d\theta$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\tan^2\theta}}$$

$$\boxed{\sec^2\theta = 1 + \tan^2\theta}$$

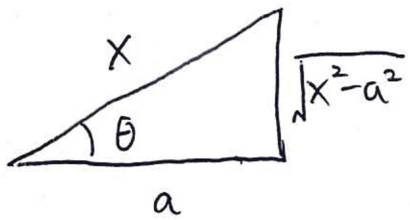
$$= \int \sec\theta\, d\theta$$

$$\text{*} = \ln|\sec\theta + \tan\theta| + C$$

$$\text{**} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$x = a\sec\theta = \dfrac{a}{\cos\theta}$

$\cos\theta = \dfrac{a}{x}$ $\quad$ **



* Ex $\displaystyle\int \sec\theta\, d\theta = ?$ (Will discuss later)

Hint 1: $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\cos\theta}{\cos^2\theta}$

Hint 2: $\dfrac{1}{1-y^2} = \dfrac{1}{2}\left(\dfrac{1}{1-y} + \dfrac{1}{1+y}\right)$

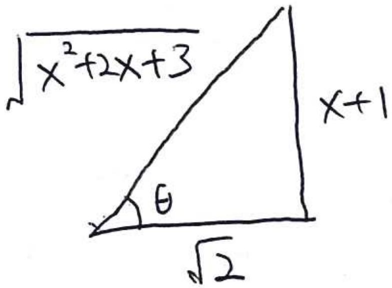
eg $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+3}}$

$$= \int \frac{dx}{\sqrt{(x+1)^2+2}} \qquad \text{Let } x+1 = \sqrt{2}\tan\theta$$

$$dx = \sqrt{2}\sec^2\theta \, d\theta$$

$$= \int \frac{\sqrt{2}\sec^2\theta \, d\theta}{\sqrt{2\tan^2\theta + 2}}$$

$$= \int \frac{\sqrt{2}\sec^2\theta \, d\theta}{\sqrt{2}\sec\theta}$$



$$= \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+2x+3} + x+1}{\sqrt{2}}\right| + C$$

t-formula

Let $t = \tan\frac{x}{2}$. Then

$$\sin x = \frac{2t}{1+t^2} \qquad \csc x = \frac{1+t^2}{2t}$$

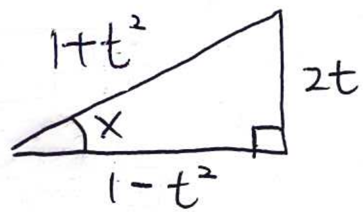
$$\cos x = \frac{1-t^2}{1+t^2} \qquad \sec x = \frac{1+t^2}{1-t^2}$$

$$\tan x = \frac{2t}{1-t^2} \qquad \cot x = \frac{1-t^2}{2t}$$

$$dx = \frac{2}{1+t^2}\,dt$$

Pf $\displaystyle \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$

$\Rightarrow$



$$\sqrt{(1-t^2)^2+(2t)^2}$$
$$= \sqrt{1-2t^2+t^4+4t^2}$$
$$= 1+t^2$$

Also, $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$$= \frac{1}{2}(1+t^2)\,dx \Rightarrow dx = \frac{2}{1+t^2}\,dt$$

t-formula is useful for rational functions of trig functions, ie. $\dfrac{\text{polynomial in trig functions}}{\text{polynomial in trig functions}}$

eg

$$\int \csc x \, dx$$

$$= \int \frac{1}{\sin x} dx$$

$$= \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{1}{t} dt$$

$$= \ln|t| + C$$

$$= \ln\left|\tan\frac{x}{2}\right| + C$$

eg $\displaystyle\int \frac{1}{1-\cos x} dx$

$$= \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt$$

$$= \int \frac{2\,dt}{1+t^2-(1-t^2)}$$

$$= \int \frac{1}{t^2} dt$$

$$= -\frac{1}{t} + C$$

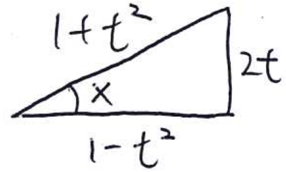
$$= -\cot\frac{x}{2} + C$$

eg $\displaystyle\int \frac{dx}{4\sin x + 3\cos x + 3}$

Let $t = \tan\frac{x}{2}$

$$dx = \frac{2dt}{1+t^2}$$

$$dt = \left(\sec^2\frac{x}{2}\right)\left(\frac{1}{2}\right) dx = \frac{1+t^2}{2} dx$$



$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \int \frac{dx}{4\sin x + 3\cos x + 3} = \int \frac{\frac{2dt}{1+t^2}}{4\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right) + 3}$$

$$= \int \frac{2dt}{8t + 3 - 3t^2 + 3 + 3t^2}$$

$$= \int \frac{dt}{4t+3}$$

$$= \frac{1}{4}\int \frac{d(4t+3)}{4t+3}$$

$$= \frac{1}{4}\ln|4t+3| + C$$

$$= \frac{1}{4}\ln\left|4\tan\frac{x}{2} + 3\right| + C$$